The Tail of the Colored Jones Polynomial

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Table of contents

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- If K is the unknot $J_{N,K}(q) = 1$.

Kauffman Bracket Relations

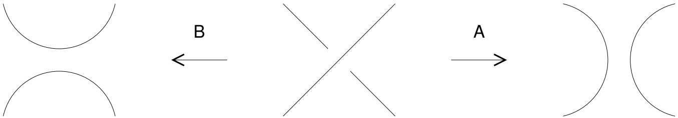

Figure: A and B smoothings

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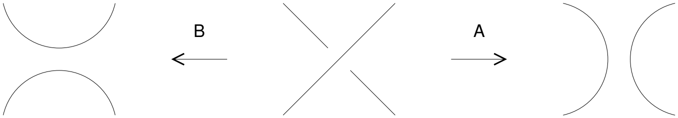

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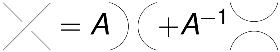
- 

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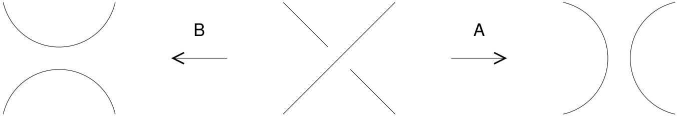

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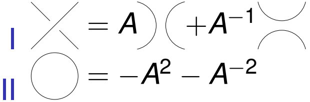

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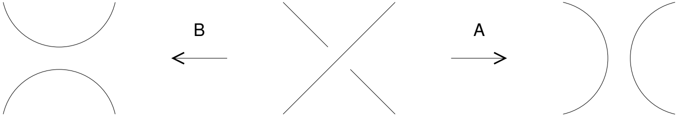

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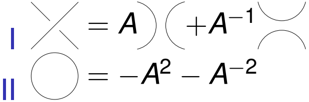


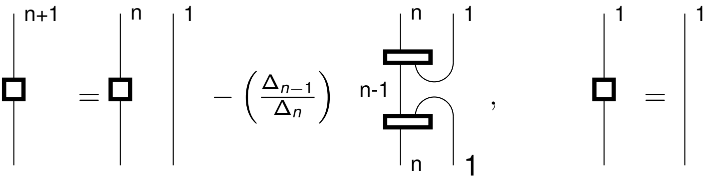
Definition (Kauffman Bracket Skein Module)

For a 3-manifold M, a ring R (usually $\mathbb{Z}[A, A^{-1}]$ or $\mathbb{Q}(A)$), and an invertible element A of R:

$$S(M; R, A) := R\{\text{framed links in } M\}/(I, II)$$

Jones Wenzl Idempotent

$$\Delta_n := (-1)^n \frac{A^{2(n+1)} - A^{-2(n+1)}}{A^2 - A^{-2}}$$



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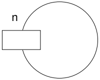
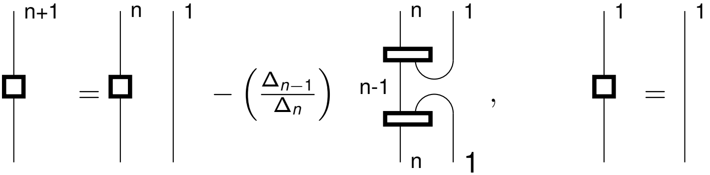


$= \Delta_n$



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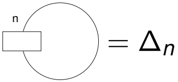

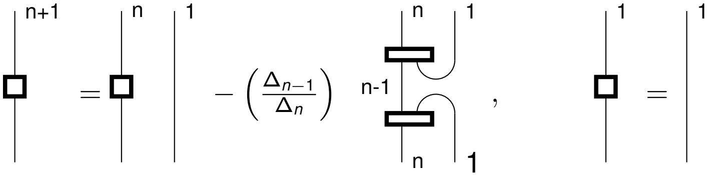

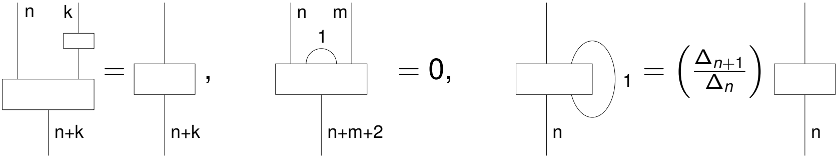

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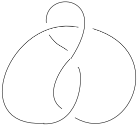
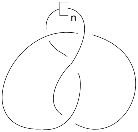
If $L =$ , then define

$$\tilde{J}_{n,L}(A) = $$ 

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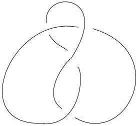
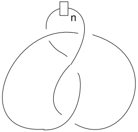
If $L =$ , then define

$$\tilde{J}_{n,L}(A) =$$ 

$$\in S(\mathbb{R}^3; \mathbb{Z}[A, A^{-1}], A) \cong \mathbb{Z}[A, A^{-1}]$$

$$J_{n+1,L}(q) := \left. \frac{\tilde{J}_{n,L}(A)}{\Delta_n} \right|_{A=q^{-1/4}}$$

For L an unframed link, give it the 0-framing. Changing framing changes the values of $\tilde{J}_{n,L}(A)$ by multiplying by $\pm A$ to some power.

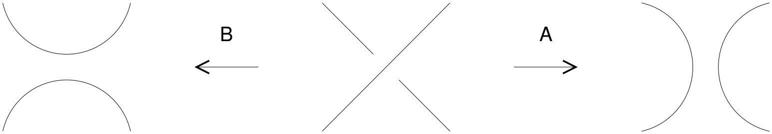
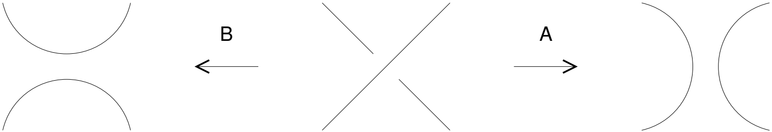


Figure: A and B smoothings



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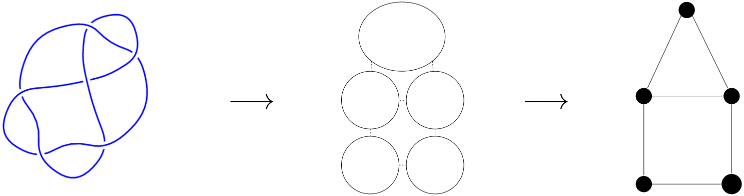
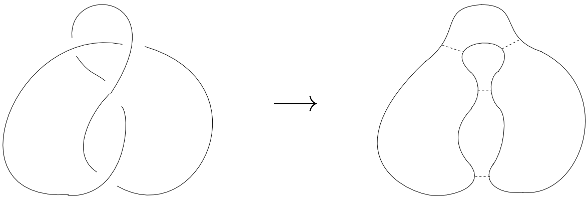
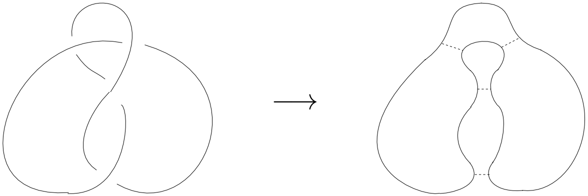
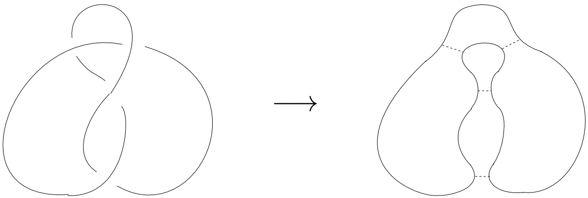


Figure: The All-B state graph G_B of 6_2





- A diagram is A-adequate if the all-A state graph has no loops.
- A diagram is B-adequate if the all-B state graph has no loops.
- A diagram is adequate if it is both A and B adequate.
- All reduced alternating link diagrams are adequate.



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Remark: The all-A and all-B state graphs of a reduced alternating diagram are the same as the Tait graphs (checkerboard graphs).

Reduced Graphs

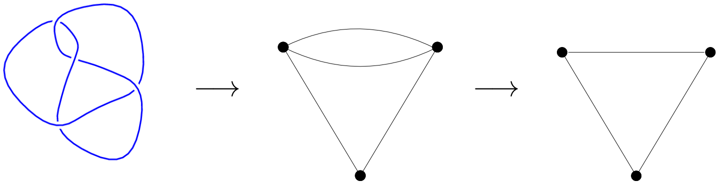

Figure: The reduced graph G'_A of 4_1

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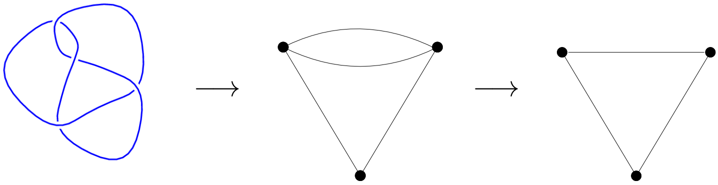

Figure: The reduced graph G'_A of 4_1

To construct the reduced all-A graph G'_A, remove multiple edges from G_A.

An Equivalence Relation on Laurent Series

Definition

For two Laurent series $P_1(q)$ and $P_2(q)$ we define

$$P_1(q) \doteq_n P_2(q)$$

if the first n coefficients agree up to a universal sign.

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For example $-q^{-4} + 2q^{-3} - 3 + 11q \doteq_5 1 - 2q + 3q^4$.

The Head and Tail

Definition

The tail of the colored Jones polynomial of a link L – if it exists – is a series $T_L(q)$, with

$$T_L(q) \doteq_N J_{L,N}(q), \text{ for all } N$$

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The head of the colored Jones polynomial of a link L – if it exists – is a series $H_L(q)$, with

$$H_L(q) \doteq_N J_{L,N}(q^{-1}) = J_{\bar{L},N}(q), \text{ for all } N$$

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Remark: $T_L(q)$ exists $\iff J_{N,L}(q) \doteq_N J_{N+1,L}(q)$

Example

Here is a table of colored Jones polynomials for the knot 6_2:

$$
\begin{array}{ll}
N=2: & 1-2q+2q^2-2q^3+2q^4-q^5+q^6 \\
N=3: & 1-2q+0q^2+4q^3-5q^4+6q^6+\dots \\
N=4: & 1-2q+0q^2+2q^3+q^4-4q^5-2q^6+\dots \\
N=5: & 1-2q+0q^2+2q^3-q^4+2q^5-6q^6+\dots \\
N=6: & 1-2q+0q^2+2q^3-q^4+0q^5-2q^7+q^8+\dots \\
N=7: & 1-2q+0q^2+2q^3-q^4+0q^5-2q^6+4q^7-3q^8+\dots
\end{array}
$$

$$T_{6_2}(q)=1-2q+0q^2+2q^3-q^4+0q^5-2q^6+\dots$$

Theorem

Theorem (A)

If L has an A-adequate diagram, then $T_L(q)$ *exists.*
If L has an B-adequate diagram, then $H_L(q)$ *exists.*

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If L has an A-adequate diagram, then $T_L(q)$ exists.
If L has an B-adequate diagram, then $H_L(q)$ exists.

Corollary

If L is an alternating link, then $T_L(q)$ and $H_L(q)$ exists.

The Main Lemma

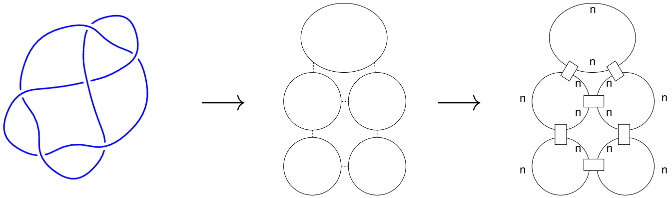


Figure: The diagram $S_B^{(n)}$ for 6_2

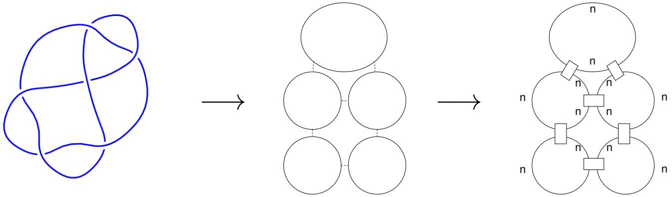
Theorem (A., Dasbach)

If D is a reduced alternating diagram, then

$$\tilde{J}_{n,L}(A) \doteq_{4(n+1)} S_B^{(n)}$$

Remark: $S_B^{(n)} \in \mathbb{Q}(A) \hookrightarrow \mathbb{Q}[[A]][A^{-1}]$

The Main Lemma



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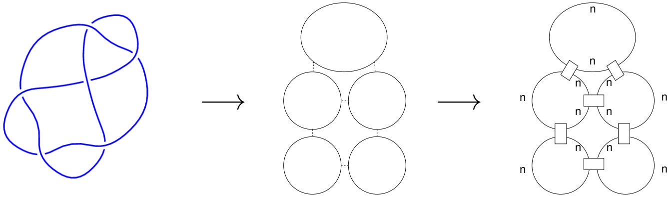

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Corollary

$T_L(q)$ only depends on G'_A

This induces a map T : Planar Graphs $\to \mathbb{Z}[[q]]$

Brief Sketch of Proof

Theorem

If D is a B-adequate diagram for a link L with corresponding $S_B^{(n)}$, then

$$S_B^{(n)} \doteq_{4(n+1)} S_B^{(n+1)}$$

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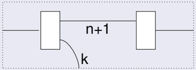
Lemma



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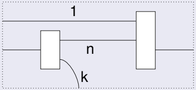
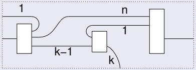


$$-\left(\frac{\Delta_{k-1}}{\Delta_{n+k}}\right)$$



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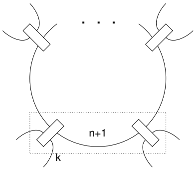

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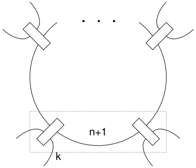
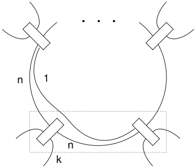


$\doteq 4(n+1)$



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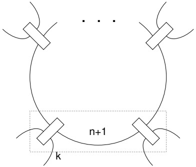

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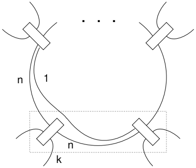

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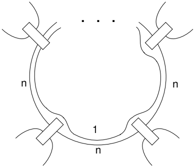

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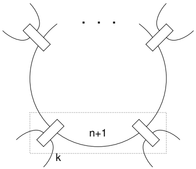

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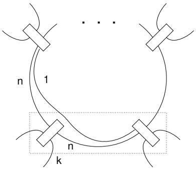

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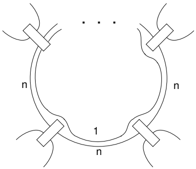

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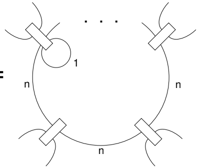

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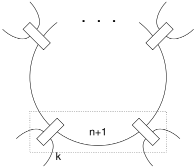

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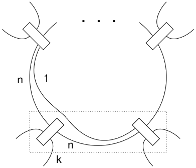

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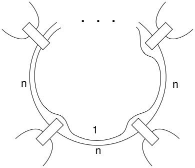

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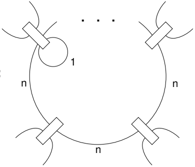
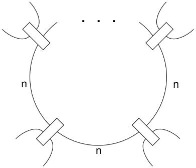


$= \left(\dfrac{\Delta_{n+k+1}}{\Delta_{n+k}}\right)$



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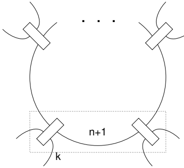

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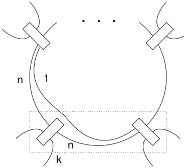

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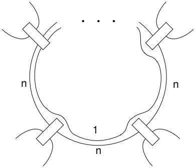
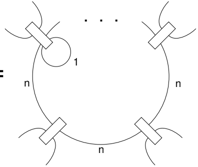
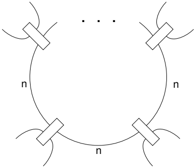


$=$



$= \left(\frac{\Delta_{n+k+1}}{\Delta_{n+k}} \right)$



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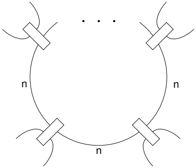

Torus Knots

Here is a table of the first N terms of $\pm q^{s_N} J_{N,K}(q)$ for the knot $\bar{8}_{19}$, the (negative) $(3,4)$ torus knot:

$$
\begin{aligned}
N = 2 &: \quad 1 \\
N = 3 &: \quad 1 - q \\
N = 4 &: \quad 1 - q^2 - q^3 \\
N = 5 &: \quad 1 - q \\
N = 6 &: \quad 1 - q^2 - q^3 \\
N = 7 &: \quad 1 - q - q^6
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\end{aligned}$$

Theorem (A., Dasbach)

Let $p > m$. A (m,p)-torus knot has one head and one tail if $m = 2$ and two heads and one tail if $m > 2$. The two heads correspond to even or odd N.

Proof: Morton's formula

Theorem (A., Dasbach)

For $k > 0$, and C_n, the cycle on n vertices,

- $T(C_{2k+1}) = f(-q^{2k}, -q)$
  (From Morton's formula for $(2, 2k+1)$ torus knots)
- $T(C_{2k}) = \Psi(q^{2k-1}, q)$
  (From Hikami's formula for $(2, 2k)$ torus links)

$$f(a, b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}$$

$$\Psi(a, b) := \sum_{k=0}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2} - \sum_{k=1}^{\infty} a^{k(k-1)/2} b^{k(k+1)/2}$$

Products

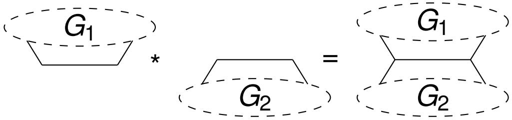
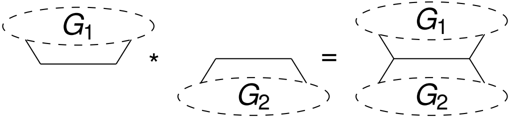


Figure: Product of two checkerboard graphs

The previously defined function T respects this product.

Products



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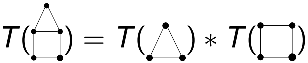
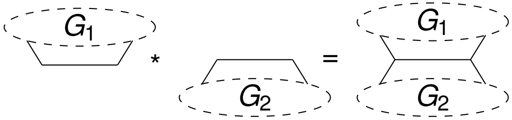


Products



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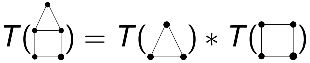

Thus,

$$T_{\bar{6}_2}(q) = f(-q^2, -q) * \Psi(q^3, q)$$

Products

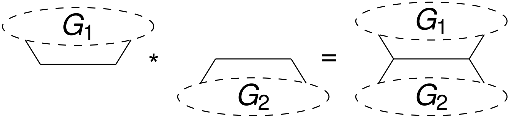

Figure: Product of two checkerboard graphs

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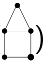
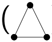
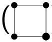
$T($$) = T($$) * T($$)$

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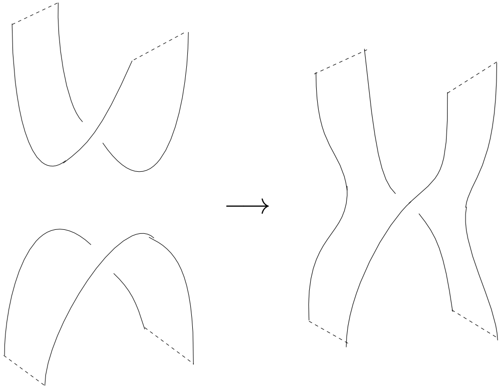
The tail of any 2-bridge knot is a product of Ramanujan theta functions and false theta functions

What is this product on knots?

What is this product on knots?

Answer: Plumbing



Theorem (A., Dasbach)

For any A-adequate link L, there is an alternating link L_0 with

$$T_L(q) = T_{L_0}(q)$$

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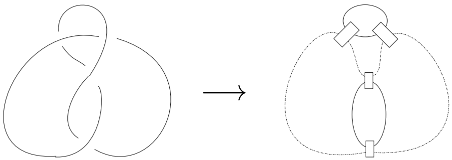
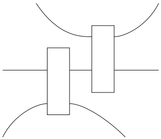
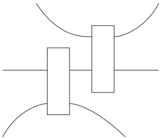
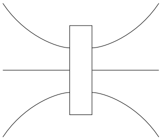
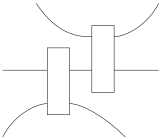






$\doteq_{4(n+1)}$

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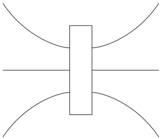
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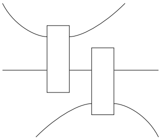
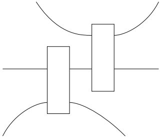
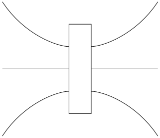
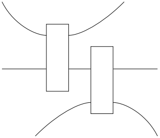


 $\doteq_{4(n+1)}$  $\doteq_{4(n+1)}$ 

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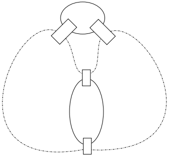
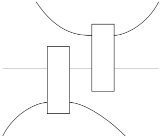
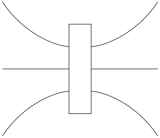
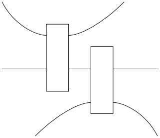


 $\doteq_{4(n+1)}$  $\doteq_{4(n+1)}$ 

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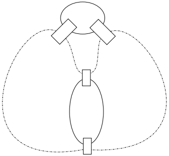
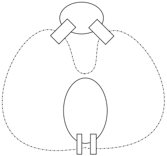
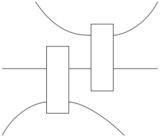
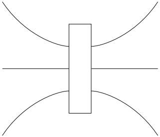
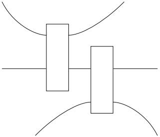
 $\doteq_{4(n+1)}$ 

 $\doteq_{4(n+1)}$  $\doteq_{4(n+1)}$ 

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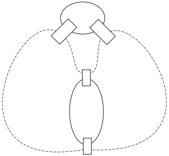
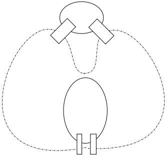
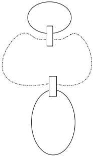
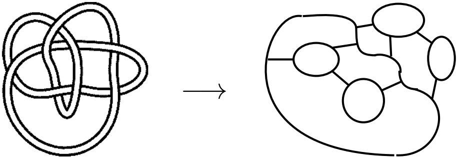
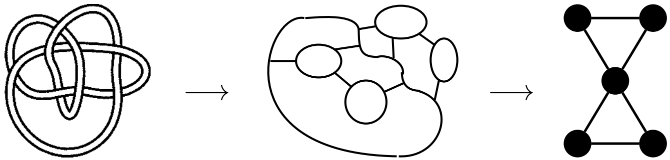
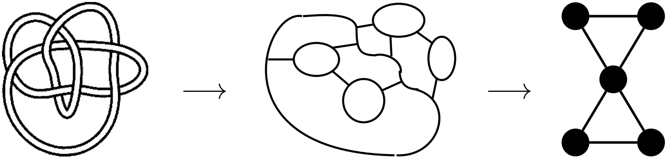
 $\doteq_{4(n+1)}$  $\doteq_{4(n+1)}$ 







Thus $T_{8_{21}}(q) = \left[f(-q^2, -q)\right]^2$.

Theorem (Dasbach, Lin)

If K is an A-adequate knot, the second coefficient of the $T_K(q)$ is $-\beta_1(G'_A)$, that is the first Betti number of the reduced all-A graph; $\beta_1(G'_A) = e - v + 1$.

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Corollary (A., Dasbach)

If K is an A-adequate knot, then $T_K(q) = 1$ if and only if G'_A is a tree.

Theorem (Dasbach, Lin)

If K is an A-adequate knot, then

$$T_K(q) = 1 - \beta q + \left(\binom{\beta}{2} - \tau \right) q^2 + \dots$$

where $\beta = \beta_1(G'_A)$ and τ is the number of triangles (3-cycles) in G'_A

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Vague Conjecture

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Theorem (Garoufalidis, Norin, Vuong)

The conjecture is true for $n = 4$

Theorem (A)

If K is an A-adequate knot for which the shortest cycle in G_A' is ℓ, then

$$T_K(q) = 1 - \beta q + \binom{\beta}{2} q^2 - \binom{\beta}{3} q^3 + \cdots + (-1)^{\ell-1}\left(\binom{\beta}{\ell-1} - c_\ell\right) q^{\ell-1} + O(\ell)$$

where c_ℓ is the number of ℓ cycles.

Theorem (A)

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$$T_K(q) \quad = \quad (1-q)^{\beta} + (-1)^{\ell} c_{\ell} q^{\ell-1} + O(\ell)$$

where c_{ℓ} is the number of ℓ cycles.

Thank You