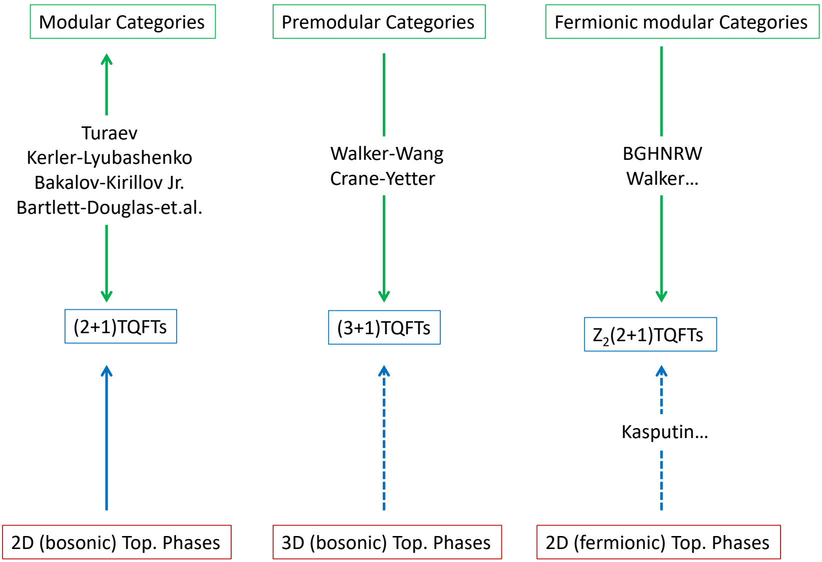
Classification of Pre-modular Categories

Eric Rowell Texas A&M

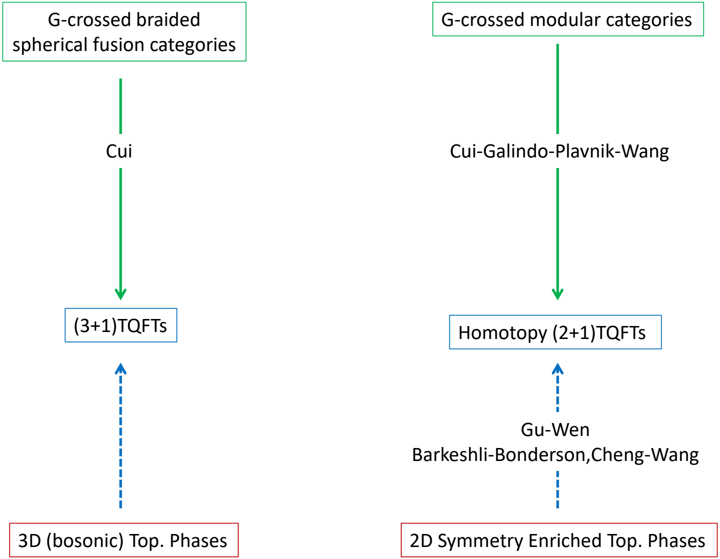
The Ohio State U., December 2016

Supported by USA NSF grant DMS1410144

joint work with Ng, Bruillard, Wang, Galindo, Plavnik... in 10+ papers



Modular Categories
Premodular Categories
Fermionic modular Categories
Turaev
Kerler-Lyubashenko
Bakalov-Kirillov Jr.
Bartlett-Douglas-et.al.
Walker-Wang
Crane-Yetter
BGHNRW
Walker...
(2+1)TQFTs
(3+1)TQFTs
Z_2(2+1)TQFTs
Kasputin...
2D (bosonic) Top. Phases
3D (bosonic) Top. Phases
2D (fermionic) Top. Phases


G-crossed braided spherical fusion categories
Cui
(3+1)TQFTs
3D (bosonic) Top. Phases
G-crossed modular categories
Cui-Galindo-Plavnik-Wang
Homotopy (2+1)TQFTs
Gu-Wen
Barkeshli-Bonderson,Cheng-Wang
2D Symmetry Enriched Top. Phases

A **pre-modular category** $\mathcal{C}$ (over $\mathbb{C}$) is

monoidal: $(\otimes, \mathbf{1})$,

semisimple: $X \cong \bigoplus_i m_i X_i$,

linear: $\mathrm{Hom}(X, Y) \in Vec_{\mathbb{C}}$,

rigid: $X^* \otimes X \mapsto \mathbf{1} \mapsto X \otimes X^*$,

finite rank: $\mathrm{Irr}(\mathcal{C}) = \{\mathbf{1} = X_0, \ldots, X_{r-1}\}$,

spherical: $theta_X : X \cong X$, $\dim(X) \in \mathbb{R}$,

braided: $c_{X,Y} : X \otimes Y \cong Y \otimes X$,

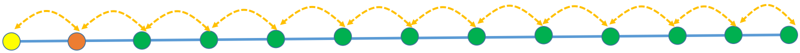
- Plus lots of compatibility...
- $\mathcal{C}$ is modular if $S_{ij} := \mathrm{Tr}_{\mathcal{C}}(c_{X_i,X_j^*} c_{X_j^*,X_i})$ is invertible.
- $\mathcal{C}$ is symmetric if $c_{X,Y} c_{Y,X} = Id_{Y \otimes X}$ for all X, Y.
- dimensions: $d_i := S_{0i}, \dim(\mathcal{C}) = \sum_i d_i^2$
- fusion (multiplicity) rules: $N_{ij}^k := \dim \mathrm{Hom}(X_i \otimes X_j, X_k)$

A Modular Category

Example

$SU(2)_{\ell-2}$ from $\text{Rep}(U_q\mathfrak{sl}_2)$ at $q = e^{\pi i/\ell}$

- simple objects $\{X_0 = \mathbf{1}, X_1, \ldots, X_{\ell-2}\}$
- $S_{ij} = \frac{\sin(\frac{(i+1)(j+1)\pi}{\ell})}{\sin(\frac{\pi}{\ell})}$ and $d_i = [i+1]_q := \frac{q^{i+1}-q^{-i-1}}{q-q^{-1}}$
- $\theta_j = e^{\frac{\pi i (j^2+2j)}{2\ell}}$
- $X_1 \otimes X_k \cong X_{k-1} \oplus X_{k+1}$ for $1 \leq k \leq \ell - 3$



The Müger Center

Definition

Let $\mathcal{C}$ be a braided monoidal category. $X \in \mathcal{C}$ is **transparent** if $c_{X,Y}c_{Y,X} = Id_{Y\otimes X}$ for all $Y \in \mathcal{C}$. The transparent objects form a full (symmetric) subcategory called the **Müger center**, denoted $\mathcal{C}'$.

Definition

More generally, for $\mathcal{D} \subset \mathcal{C}$ pre-modular the **relative (Müger) center** $Z_{\mathcal{C}}(\mathcal{D}) := \{Y \in \mathcal{C} : c_{X,Y}c_{Y,X} = Id_{Y\otimes X}, X \in \mathcal{D}\}$

Theorem (Müger)

$\mathcal{C}$ pre-modular: X, Y simple $c_{X,Y}c_{Y,X} = Id_{Y\otimes X}$ iff $S_{X,Y} = d_X d_Y$.

(super-)Modularity

Theorem (Bruguieres)

$\mathcal{C}$ pre-modular is modular iff $\mathcal{C}' \cong Vec = \langle \mathbf{1} \rangle$.

Theorem (Deligne)

Symmetric pre-modular categories are equivalent to $\mathrm{Rep}(G, z)$*: super-representations of G, where $z \in Z(G)$, $z^2 = 1$ determines braiding.*

Example

The *unitary* pre-modular category $sVec \cong \mathrm{Rep}(\mathbb{Z}_2, -1)$ is symmetric. Two simples: $\mathbf{1}, f$ with $c_{f,f} = -Id_{f \otimes f}$ and $\theta_f = -1$.

Definition (KEY)

Unitary pre-modular $\mathcal{C}$ is **super-modular** if $\mathcal{C}' \cong sVec$. Generally, if $X \in \mathcal{C}$ simple has $\langle X \rangle \cong sVec$, X is a **fermion**, while if $\langle X \rangle \cong \mathrm{Rep}(\mathbb{Z}_2)$, X is a **boson**.

A super-modular category

Example

PSU(2)$_{4m+2}$: "integer half" of SU(2)$_{4m+2}$ is **super-modular**

- simple objects $\{Y_0 = \mathbf{1}, Y_1 = X_2, \ldots, Y_{2m+1} = X_{4m+2}\}$
- $S_{ij} = \frac{\sin(\frac{(2i+1)(2j+1)\pi}{(4m+4)})}{\sin(\frac{\pi}{(4m+4)})}$ and $\theta_j = e^{\frac{\pi i (j^2+j)}{2m+2}}$
- $Y_1 \otimes Y_k \cong Y_{k-1} \oplus Y_{k+1} \oplus Y_k$ for $k \ll \infty$

- Notice: $Y_{2m+1}^{\otimes 2} = \mathbf{1}$, $\theta_{2m+1} = -1$ and $S_{j,2m+1} = d_j d_{2m+1} \forall j$
- In fact: $\text{PSU}(2)'_{4m+2} = \langle Y_{2m+1} \rangle \cong sVec$.

Top 10 Reasons to Like Modular Categories

Let $\mathcal{C}$ be a modular category of rank r, with N_{ij}^k the fusion coefficients. Define $d_j := S_{0j}$, $T_{ij} := \delta_{ij}\theta_j$, $D^2 := \sum_j d_j^2$

1-3 $S = S^t$, $SS^\dagger = D^2 Id$, $ord(T) = N < \infty$

4 S, T give proj. rep. of $\mathrm{SL}(2,\mathbb{Z})$, factors over $\mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z})$

5 $N_{ij}^k = \sum_a \frac{S_{ia}S_{ja}\overline{S_{ka}}}{D^2 d_a}$

6 $\theta_i\theta_j S_{ij} = \sum_a N_{i^*j}^k d_k \theta_k$.

7 $\nu_n(k) := \frac{1}{D^2}\sum_{i,j} N_{ij}^k d_i d_j \left(\frac{\theta_i}{\theta_j}\right)^n \in \mathbb{Z}[\zeta_N]$ and $\nu_2(k) \in \{0, \pm 1\}$

8 $\mathbb{Q}(S) \subset \mathbb{Q}(T)$, $\mathrm{Aut}_{\mathbb{Q}}\,\mathbb{Q}(S) \subset \mathfrak{S}_r$, $\mathrm{Aut}_{\mathbb{Q}(S)}\,\mathbb{Q}(T) \cong (\mathbb{Z}_2)^k$.

9 Prime (ideal) divisors of $\langle D^2 \rangle$ and $\langle N \rangle$ coincide in $\mathbb{Z}[\zeta_N]$.

10 There are finitely many modular categories (of fixed rank r).

Most of these are false (or nonsense) for pre-modular categories.

De-equivariantization

Given a Tannakian subcategory $\text{Rep}(G) \cong \mathcal{D} \subset \mathcal{C}$ of a fusion category $\mathcal{C}$ one may construct the de-equivariantization $\mathcal{C}_G$ of $Fun(G)$-modules, where $Fun(G) \in \text{Rep}(G)$ is the **regular algebra** and G is a finite group.

- $\mathcal{C}_G$ is G-graded.
- $\dim \mathcal{C}_G = \dim(\mathcal{C})/|G|$
- If $\mathcal{C}$ is braided and $\mathcal{D} \subset \mathcal{C}'$ then $\mathcal{C}_G$ is braided.

Lemma

Let $\mathcal{C}$ be a pre-modular category, and $\text{Rep}(G) \cong \mathcal{T} \subset \mathcal{C}'$ be the maximal, Tannakian, central subcategory. Then $\mathcal{C}_G$ is either modular (if $\mathcal{T} = \mathcal{C}'$) or super-modular.

Remark

The process can be reversed via **equivariantization**: $\mathcal{D} \to \mathcal{D}^G$. We therefore reduce the classification of pre-modular categories to modular/super-modular.

Sources of Modular Categories

- Pointed: $\mathcal{C}(A,q)$, A finite abelian group, q non-degenerate quad. form on A.
- Quantum groups: "purifications" of $\text{Rep}(U_q\mathfrak{g})$ at $q = e^{\pi i/\ell}$ with some restrictions on ℓ.
- Group-theoretical: $\text{Rep}(D^\omega G)$ finite group G, 3-cocycle ω. $\mathcal{D} \subset \text{Rep}(D^\omega G)$.
- Drinfeld center: $\mathcal{Z}(\mathcal{D})$ for spherical fusion category $\mathcal{D}$ (say, from a subfactor....)
- Deligne products: $\mathcal{C} \cong \mathcal{D}_1 \boxtimes \mathcal{D}_2$. If not of this form, **prime**.

A Sampler of Classifications of Modular Categories

Definition

$\mathcal{C}$ is **weakly integral** if $\dim(\mathcal{C}) \in \mathbb{N}$, and **integral** if $\dim(X) \in \mathbb{N}$

The following modular categories are classified:

- Rank≤ 5
- Weakly Integral, Rank≤ 7
- $\dim(\mathcal{C})$ odd, Rank≤ 11

Let p, q, r be primes and m a square-free integer.

- $\dim(\mathcal{C}) = p^2 m$ or $p^3 m$, $\gcd(m, p) = 1$ (Weakly Integral).
- Integral: $\dim(\mathcal{C}) = mp^n$, $(p, m) = 1, n \leq 5$,
  $\dim(\mathcal{C}) \in \{pq^2, pqr, p^n\}$ pq^n, $p < q$

Conjecture

Every odd-dimensional modular category is group-theoretical.

Classifications are due to many different authors...

Remark

Some classifications are more explicit than others:

- Up to equivalence
- A possibly redundant list
- A list of possible pairs (S, T)
- Fusion rules
- An acceptable characterization (e.g. "all are group-theoretical")

A Worked Example

To classify modular categories of dimension $p^3 m$ where p is prime, m is square-free and $\gcd(m, p) = 1$.

- If $\mathcal{C}$ is pointed, $\mathcal{C} \cong \mathcal{C}(A, q)$ victory!
- Suppose $\mathcal{C}$ is not pointed. Then $p = 2$.
- If $\mathcal{C}$ is not prime, then $\mathcal{C}$ has a pointed Deligne factor (can be dealt with by induction etc.)
- If $\mathcal{C}$ is prime, then $\mathcal{C}$ has the same fusion rules as $SO(2p)_2$, and simple objects have dimension $1, 2$ or $\sqrt{p}$.
- Let $\mathcal{D}$ be any modular category with fusion rules like $SO(2N)_2$, N odd ($\mathcal{D}$ is *even metaplectic*). Then $\mathcal{D}$ has a boson $\langle b \rangle \cong \text{Rep}(\mathbb{Z}_2)$.
- De-equivariantize: $(\mathcal{D}_{\mathbb{Z}_2})_0 \cong \mathcal{C}(\mathbb{Z}_{2N}, q)$ is modular. $\mathbb{Z}_2$-Extensions/equivariantizations known. victory!

Sources of Super-modular Categories

- **split super-modular**: $\mathcal{C} \boxtimes sVec$, $\mathcal{C}$ modular.
- Let $\mathcal{C}$ be modular, with $f \in \mathcal{C}$ a fermion. $Z_{\mathcal{C}}(\langle f \rangle) \subset \mathcal{C}$ is super-modular and has $\dim Z_{\mathcal{C}}(\langle f \rangle) = \dim \mathcal{C}/2$

Conjecturally, these are all of them:

Conjecture (Davydov-Nikshych-Ostrik)

Every super-modular category $\mathcal{D}$ has $\mathcal{D} \subset \mathcal{C}$ where $\mathcal{C}$ is modular and $\dim \mathcal{C} = 2 \dim \mathcal{D}$.

This is a special case of a (false) conjecture of Müger, who calls such $\mathcal{C}$ a **minimal modular extension**.

16-fold way

Example (Kitaev)

There are exactly 16 modular $\mathcal{C} \supset sVec$ with $\dim \mathcal{C} = 4 = 2 \dim sVec$: $SO(N)_1$ for $1 \leq N \leq 16$.

Remark

$N = 1$ is Ising, $N = 2$ is $\mathcal{C}(\mathbb{Z}_4, q)$ $N = 16$ is Toric Code,...

This is a general phenomena (Tian-Kong-Wen, generalizing our result):

Theorem

If super-modular $\mathcal{D}$ has one minimal modular extension, it has exactly 16.

Why are super-modular categories nice?

- They are even rank: $-\otimes f$ is fixed-point-free.
- $\nu_2(k) := \frac{1}{D^2}\sum_{i,j} N_{ij}^k d_i d_j \left(\frac{\theta_i}{\theta_j}\right)^2 \in \{0, \pm 1\}$.
- $S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \hat{S},\ T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \hat{T}$ (not unique)
- $\hat{S}, \hat{T}^2$ give proj. rep of $\Gamma_0(2) \subset SL(2, \mathbb{Z})$.
- A version of Verlinde formula using $\hat{S}$ holds.
- A classification of rank≤ 6 is known.

Some Examples

The following are modular categories with a fermion:

- $SU(4k+2)_{4m+2}$
- $SO(2k+1)_{2m+1}$
- $Sp(2r)_m$ with $rm = 2 \pmod 4$,
- $SO(2r)_m$ with $r = 2 \pmod 4$ and $m = 2 \pmod 4$,
- $(E_7)_{4m+2}$.

Why Can't we de-equivariantize $\mathcal{C}$ by $sVec = \langle f \rangle$?

Answer

We can, but the result $\mathcal{Q} = \mathcal{C}_{sVec}$ is super-fusion: $\mathrm{Hom}_{\mathcal{Q}}(X, Y) \in sVec$. We call them **fermionic quotients**

The idea: $\mathrm{Hom}_{\mathcal{Q}}(X, Y) := (\mathrm{Hom}(X, Y), \mathrm{Hom}(X, f \otimes Y))$.

Example ($(PSU(2)_6)_{sVec}$)

1. $PSU(2)_6$ Simple objects: $[\mathbf{1}, Y_1, Y_2, Y_3 = f]$
2. $f \otimes Y_1 = Y_2$, $f \otimes f = \mathbf{1}$, so $Y_1 \leftrightsquigarrow Y_2$ and $\mathbf{1} \leftrightsquigarrow f$
3. $\hat{S} = \begin{pmatrix} 1 & 1+\sqrt{2} \\ 1+\sqrt{2} & -1 \end{pmatrix}$, $\hat{T} = Diag(1, i)$
4. $N_1 = \begin{pmatrix} (0,0) & (1,0) \\ (1,0) & (1,1) \end{pmatrix}$ where $(a,b)(c,d) = (ac + bd, bc + ad)$.

Conjecture

(a) For any fermionic quotient, $\hat{S}$ and $\hat{T}^2$ generate a finite group, (but $\hat{S}$, $\hat{T}$ do not).

(b) $\mathcal{Q}$ is pure-braided, and image of $\mathcal{P}_n$ is finite iff $\dim(\mathcal{Q})^2 \in \mathbb{Z}$.

(c) There are finitely many fermionic quotients of a given rank k, (and therefore finitely many pre-modular categories of rank r).

Thank you!